

**ON THE FINE SPECTRUM OF THE SECOND ORDER  
DIFFERENCE OPERATOR OVER THE SEQUENCE SPACES  
 $\ell_p$  AND  $bv_p$ ,  $(1 < p < \infty)$ .**

VATAN KARAKAYA\*, MANAF DZH. MANAFOV, AND NECIP ŞİMŞEK

ABSTRACT. In general, it is well known the behaviors of the symmetric tri-band matrices on the Hilbert spaces. But the symmetric tri-band matrices have different the behavior on the Banach spaces. The main purpose of this work is to determine the fine spectra of the operator  $U(s, r, s)$  defined by symmetric tri-band matrix over the sequence spaces  $\ell_p$  and  $bv_p$ .

## 1. INTRODUCTION

As it is well known, the matrices play an important role in operator theory. The spectrum of an operator generalizes the notion of eigenvalues for matrices. In the calculation of the spectrum of an operator over a Banach space, we mostly deal with three disjoint parts of the spectrum, which are the point spectrum, the continuous spectrum and the residual spectrum. The determination of these three parts of the spectrum of an operator is called the fine spectra.

Over the years and different names the spectrum and fine spectra of linear operators defined by some particular limitation matrices over some sequence spaces have been studied.

In the existing literature, there are many papers concerning the spectrum and the fine spectra of an operator over different sequence spaces. For example; Gonzales[14], Akhmedov and Başar[2] computed the fine spectra of the Cesáro operator over the sequence spaces  $\ell_p$  and  $c_0$  respectively. Also Reade[19] and Okutoyi[18] examined the spectrum of the Cesáro operator over the spaces  $c_0$  and  $bv$  respectively. Later on Akhmedov and Başar[1, 3] also determined the spectrum of the Cesáro operator and the fine spectrum of the difference operator  $\Delta$  over the sequence space  $bv_p$   $(1 < p < \infty)$ . Also, in [21], Wenger studied the fine spectra of Hölder summability operators over the space  $c$  and Rhoades[20] extended this result to the weighed mean method.

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Recently, Karakaya and Altun([17], [5]) computed respectively the fine spectras of the upper triangular double-band matrices and the lacunary matrices as an operator over the sequence spaces  $c_0$  and  $c$ , which are defined below.

Further informations on the spectrum and fine spectra of different operators over some sequence spaces can be found in the list of references [7], [8],[10], [11].

The main purpose of our work is to determine the fine spectra of the operator for which the corresponding matrix is the symmetrical tri-band matrix  $U(s, r, s)$  over the sequence spaces  $\ell_p$  and  $bv_p$  ( $1 < p < \infty$ ).

## 2. PRELIMINARIES AND NOTATIONS

Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  be a bounded linear operator. By  $R(T)$ , we denote the range of  $T$ , i.e.,

$$R(T) = \{y \in Y : y = Tx; x \in X\}.$$

By  $B(X)$ , we denote the set of all bounded linear operators on  $X$  into itself. If  $X$  is any Banach space and  $T \in B(X)$ , then the *adjoint*  $T^*$  of  $T$  is a bounded linear operator on the dual  $X^*$  of  $X$  defined by  $(T^*\phi)(x) = \phi(Tx)$  for all  $\phi \in X^*$  and  $x \in X$ . Let  $X \neq \{\theta\}$  be a complex normed space and  $T : \mathcal{D}(T) \rightarrow X$  be a linear operator with domain  $\mathcal{D}(T) \subset X$ . By  $T$ , we associate the operator

$$T_\lambda = T - \lambda I;$$

where  $\lambda$  is a complex number and  $I$  is the identity operator on  $\mathcal{D}(T)$ . If  $T_\lambda$  has an inverse, which is linear, we denote it by  $T_\lambda^{-1}$ , that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and it is called to be the *resolvent operator* of  $T$ . Many properties of  $T_\lambda$  and  $T_\lambda^{-1}$  depend on  $\lambda$ , and spectral theory is concerned with those properties. For instance, we shall interest in the set of all  $\lambda$  in the complex plane such that  $T_\lambda^{-1}$  exists. Boundedness of  $T_\lambda^{-1}$  is another property that will be essential. We shall also ask for what all  $\lambda$  in the domain of  $T_\lambda^{-1}$  is dense in  $X$ . For investigation of  $T$ ,  $T_\lambda$  and  $T_\lambda^{-1}$ , we need some basic concepts in spectral theory which are given as follows (see [16], pp. 370-371):

Let  $X \neq \{\theta\}$  be a complex normed space and  $T : \mathcal{D}(T) \rightarrow X$  be a linear operator with domain  $\mathcal{D}(T) \subset X$ . A *regular value*  $\lambda$  of  $T$  is a complex number such that

- (R1)  $T_\lambda^{-1}$  exists,
- (R2)  $T_\lambda^{-1}$  is bounded,
- (R3)  $T_\lambda^{-1}$  is defined on a set which is dense in  $X$ .

The *resolvent set*  $\rho(T)$  of  $T$  is the set of all regular values  $\lambda$  of  $T$ . Its complement  $\sigma(T) = \mathbb{C} \setminus \rho(T)$  in the complex plane  $\mathbb{C}$  is called the *spectrum* of  $T$ . Furthermore, the spectrum  $\sigma(T)$  is partitioned into three disjoint sets as follows: The *point spectrum*  $\sigma_p(T)$  is the set such that  $T_\lambda^{-1}$  does not exist. A  $\lambda \in \sigma_p(T)$  is called an

*eigenvalue* of  $T$ . The *continuous spectrum*  $\sigma_c(T)$  is the set such that  $T_\lambda^{-1}$  exists and satisfies (R3) but not (R2). The *residual spectrum*  $\sigma_r(T)$  is the set such that  $T_\lambda^{-1}$  exists but not satisfy (R3).

We shall write  $\ell_\infty$ ,  $c$  and  $c_0$  for the spaces of all bounded, convergent and null sequences, respectively. The sequence spaces  $\ell_p$  and  $bv_p$  are defined by

$$\begin{aligned}\ell_p &= \{x \in \omega : \sum_k |x_k|^p < \infty\} \\ bv_p &= \{x \in \omega : \sum_k |x_k - x_{k+1}|^p < \infty\}.\end{aligned}$$

Let  $\mu$  and  $\gamma$  be two sequence spaces and  $A = (a_{nk})$  be an infinite matrix of real or complex numbers  $a_{nk}$ , where  $n, k \in \mathbb{N} = \{0, 1, 2, \dots\}$ . Then, we say that  $A$  defines a matrix mapping from  $\mu$  into  $\gamma$ , and we denote it by writing  $A : \mu \rightarrow \gamma$ , if for every sequence  $x = (x_k) \in \mu$  the sequence  $Ax = \{(Ax)_n\}$ , the  $A$ -transform of  $x$ , is in  $\gamma$ ; where

$$(2.1) \quad (Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}).$$

By  $(\mu : \gamma)$ , we denote the class of all matrices  $A$  such that  $A : \mu \rightarrow \gamma$ . Thus,  $A \in (\mu : \gamma)$  if and only if the series on the right side of (2.1) converges for each  $n \in \mathbb{N}$  and every  $x \in \mu$ , and we have  $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \gamma$  for all  $x \in \mu$ .

The symmetrical tri-band matrix used in our work is of the following form:

$$U(s, r, s) = \begin{bmatrix} r & s & 0 & 0 & \dots \\ s & r & s & 0 & \dots \\ 0 & s & r & s & \dots \\ 0 & 0 & s & r & \dots \\ . & . & . & . & \ddots \end{bmatrix}$$

Now let us give some the lemmas which we need in sequel

**Lemma 2.1.** Define the sets  $D_\infty$  and  $D_q$  by

$$D_\infty = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} x_j \right| < \infty \right\}$$

and

$$D_q = \left\{ x = (x_k) \in w : \sum_k \left| \sum_{j=k}^{\infty} x_j \right|^q < \infty \right\}, \quad (1 < q < \infty).$$

Then, the sets  $D_\infty$  and  $D_q$  are the Banach spaces with the norms

$$\|a\|_{D_\infty} = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} a_j \right|$$

and

$$\|a\|_{D_q} = \left( \sum_k \left| \sum_{j=k}^{\infty} a_j \right|^q \right)^{1/q}.$$

Additionally, (i)  $D_{\infty}$  is isometrically isomorphic to  $bv_1^*$ , [15, Theorem 3.3]

(ii)  $D_q$  is isometrically isomorphic to  $bv_p^*$ , [4, Theorem 2.3]

The basis of the space  $bv_p$  is also constructed and is given by the following lemma:

**Lemma 2.2.** ([6], Theorem 3.1) Define the sequence  $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$  of the elements of the space  $bv_p$  for every fixed  $k \in \mathbb{N}$  by

$$b_n^{(k)} = \begin{cases} 0, & (n < k) \\ 1, & (n \geq k) \end{cases} \quad \text{for all } n \in \mathbb{N}.$$

Then the sequence  $\{b_n^{(k)}\}_{n \in \mathbb{N}}$  is a basis for the space  $bv_p$  and any  $x \in bv_p$  has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}$$

where  $\lambda_k = x_k - x_{k-1}$  for all  $k \in \mathbb{N}$ .

**Lemma 2.3.** ([9], p.253, Theorem 34.16) The matrix  $A = (a_{nk})$  gives rise to a bounded linear operator  $T \in B(\ell_1)$  from  $\ell_1$  to itself if and only if the supremum of  $\ell_1$  norms of the columns of  $A$  is bounded.

**Lemma 2.4.** ([9], p.245, Theorem 34.3) The matrix  $A = (a_{nk})$  gives rise to a bounded linear operator  $T \in B(\ell_{\infty})$  from  $\ell_{\infty}$  to itself if and only if the supremum of  $\ell_1$  norms of the rows of  $A$  is bounded.

**Lemma 2.5.** ([9], p.254, Theorem 34.18) Let  $1 < p < \infty$  and  $A \in (l_{\infty}, l_{\infty}) \cap (l_1, l_1)$ . Then  $A \in (l_p, l_p)$ .

**Corollary 2.6.** Let  $\mu \in \{l_p, bv_p\}$  ( $1 < p < \infty$ ).  $U(s, r, s) : \mu \rightarrow \mu$  is a bounded linear operator and  $\|U(s, r, s)\|_{(\mu, \mu)} = 2|s| + |r|$ .

### 3. THE SPECTRUM OF THE OPERATOR $U(s, r, s)$ ON THE SEQUENCE SPACE $\ell_p$ , ( $1 < p < \infty$ ).

In this section, the fine spectrum of the second order difference operator  $U(s, r, s)$  over the sequence space  $\ell_p$ , ( $1 < p < \infty$ ) have been examined. We begin with a theorem concerning the bounded linearity of the operator  $U(s, r, s)$  acting on the sequence space  $\ell_p$ , ( $1 < p < \infty$ ).

**Theorem 3.1.**  $U(s, r, s) : \ell_p \rightarrow \ell_p$  is a bounded linear operator satisfying the inequalities

$$(|r|^p + 2|s|^p)^{\frac{1}{p}} \leq \|U(s, r, s)\|_{\ell_p} \leq 2|s| + |r|.$$

*Proof.* The linearity of  $U(s, r, s)$  is trivial and so it is omitted. Let us take  $e^{(1)} = (0, 1, 0, \dots)$  in  $\ell_p$ . Then  $U(s, r, s)e^{(1)} = (s, r, s, 0, 0, \dots)$  and observe that

$$\left\| U(s, r, s)e^{(1)} \right\|_{\ell_p} = (|r|^p + 2|s|^p)^{\frac{1}{p}} \leq \|U(s, r, s)\|_{\ell_p} \left\| e^{(1)} \right\|_{\ell_p}$$

which gives the fact that

$$(3.1) \quad (|r|^p + 2|s|^p)^{\frac{1}{p}} \leq \|U(s, r, s)\|_{\ell_p}$$

for any  $p > 1$ . Now take any  $x = (x_k) \in \ell_p$  such that  $\|x\| = 1$ . Then, using Minkowski's inequality and taking  $x_{-1} = 0$ , we have

$$\begin{aligned} \|U(s, r, s)x\|_{\ell_p} &= \left( \sum_{k=0}^{\infty} |sx_{k-1} + rx_k + sx_{k+1}|^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{k=0}^{\infty} |sx_{k-1}|^p \right)^{\frac{1}{p}} + \left( \sum_{k=0}^{\infty} |rx_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=0}^{\infty} |sx_{k+1}|^p \right)^{\frac{1}{p}} \\ &= (|r| + 2|s|) \|x\|_{\ell_p}, \end{aligned}$$

which gives

$$(3.2) \quad \|U(s, r, s)\|_{\ell_p} \leq 2|s| + |r|.$$

Combining the inequalities (3.1) and (3.2) we complete the proof.  $\square$

**Theorem 3.2.**  $\sigma(U(s, r, s), \ell_p) = [r - 2s, r + 2s]$ .

*Proof.* First, we prove that  $(U(s, r, s) - \lambda I)^{-1}$  exists and in  $B(\ell_p)$  for

$$\lambda \notin \{\lambda \in \mathbb{C} : \lambda = r + 2s \cos \theta, \theta \in [0, 2\pi]\}$$

and next that the operator  $(U(s, r, s) - \lambda I)$  is not invertible for

$$\lambda \in \{\lambda \in \mathbb{C} : \lambda = r + 2s \cos \theta, \theta \in [0, 2\pi]\}.$$

Let  $\lambda \notin \sigma(U(s, r, s), \ell_p)$ . Let  $\alpha_1$  and  $\alpha_2$  be the roots of the polynomial

$$P(x) = sx^2 + (r - \lambda)x + s, \text{ with } |\alpha_2| > 1 > |\alpha_1|.$$

Solving the system of equations

$$\begin{aligned} (3.3) \quad (r - \lambda)x_1 + sx_2 &= y_1 \\ sx_1 + (r - \lambda)x_2 + sx_3 &= y_2 \\ sx_2 + (r - \lambda)x_3 + sx_4 &= y_3 \\ &\dots \end{aligned}$$

for  $x = (x_k)$  in terms of  $y = (y_k)$  gives the matrix of  $(U(s, r, s) - \lambda I)^{-1}$ . This is a non-homogenous linear recurrence relation. Using the fact that  $x, y \in \ell_p$ , for (3.3)

we can reach to a solution with generating functions (see [12]). This solution can be given by

$$(3.4) \quad x_k = \frac{1}{s(\alpha_1^2 - 1)} \sum_{n=0}^{\infty} t_{kn} y_n,$$

where

$$t_{kn} = \begin{cases} \alpha_1^{k+1-n} - \alpha_1^{k+3-n}; & \text{if } k \geq n, \\ \alpha_1^{n+1-k} - \alpha_1^{n+3-k}; & \text{if } k < n. \end{cases}.$$

Thus, we obtain that

$$\|(U(s, r, s) - \lambda I)^{-1}\|_{(\ell_1, \ell_1)} = \sup_k \sum_{n=k}^{\infty} |x_k| \leq \sup_k \left( |\alpha_1| + |\alpha_1|^{2k+3} \right) \sum_{n=0}^k |\alpha_1|^n < \infty,$$

i.e.  $(U(s, r, s) - \lambda I)^{-1} \in (\ell_1, \ell_1)$ . Similarly

$$\|(U(s, r, s) - \lambda I)^{-1}\|_{(\ell_{\infty}, \ell_{\infty})} < \infty.$$

By Lemma 2.5, we have  $(U(s, r, s) - \lambda I)^{-1} \in (l_p, l_p)$ . This shows that  $\sigma(U(s, r, s), \ell_p) \subseteq \{\lambda \in C : \lambda = r + 2s \cos \theta, \theta \in [0, 2\pi]\}$ .

Let  $\lambda \in \sigma(U(s, r, s), \ell_p)$  and  $\lambda \neq r$ . Then  $(U(s, r, s) - \lambda I)^{-1}$  exists but  $y = (1, 0, 0, \dots) \in l_p$  and  $x = (x_k)$  not in  $l_p$ , hence  $|\alpha_2| > 1 > |\alpha_1|$  is not satisfied, i.e.  $(U(s, r, s) - \lambda I)^{-1}$  is not in  $B(\ell_p)$ . If  $\lambda = r$ , then  $U(s, r, s) - \lambda I = U(s, 0, s)$ . Since  $U(s, 0, s)x = \theta$  implies  $x \neq \theta = (0, 0, 0, \dots)$ ,  $U(s, r, s) : \ell_p \rightarrow \ell_p$  is not invertible. This shows that  $\{\lambda \in C : \lambda = r + 2s \cos \theta, \theta \in [0, 2\pi]\} \subseteq \sigma(U(s, r, s), \ell_p)$ . On the other hand,  $\alpha_1 \alpha_2 = 1$ ,  $|\alpha_2| > 1 > |\alpha_1|$  is not satisfied means, the roots can be only of the form

$$\alpha_1 = \frac{1}{\alpha_2} = e^{i\theta}$$

for some  $\theta \in [0, 2\pi]$ . Then  $\frac{\lambda - r}{s} = \alpha_1 + \alpha_2 = e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ . Hence  $\lambda = r + 2s \cos \theta$ , which means  $\lambda$  can be only on the line segment  $[r - 2s, r + 2s]$ . This completes the proof.  $\square$

We should remark that the index  $p$  has different meanings in the notation of the spaces  $\ell_p$ ,  $\ell_p^* \simeq \ell_q$  with  $p^{-1} + q^{-1} = 1$  and the point spectrums  $\sigma_p(U(s, r, s), \ell_p)$ ,  $\sigma_p(U^*(s, r, s), \ell_q)$  which occur in the following theorems.

**Theorem 3.3.**  $\sigma_p(U(s, r, s), \ell_p) = \emptyset$ .

*Proof.* Let  $\lambda$  be an eigenvalue of the operator  $U(s, r, s)$ . An eigenvector  $x = (x_1, x_2, \dots) \in \ell_p$  corresponding to this eigenvalue satisfies the linear system of equations

$$(3.5) \quad \begin{aligned} rx_1 + sx_2 &= \lambda x_1 \\ sx_1 + rx_2 + sx_3 &= \lambda x_2 \\ sx_2 + rx_3 + sx_4 &= \lambda x_3 \\ &\dots \end{aligned}$$

If  $x_1 = 0$ , then  $x_k = 0$  for all  $k \in \mathbb{N}$ . Hence  $x_1 \neq 0$ . Then the system of equations turn into the linear homogeneous recurrence relation

$$x_{k+2} + qx_{k+1} + x_k = 0, \text{ for } k \geq 1,$$

where  $q = \frac{r-\lambda}{s}$ . The characteristic polynomial of the recurrence relation is

$$x^2 + qx + 1 = 0.$$

There are two cases here.

**Case 1.**  $|q| = 2$ .

Then characteristic polynomial has one root:  $\alpha = \begin{cases} 1, & \text{if } q = -2 \\ -1, & \text{if } q = 2 \end{cases}$ . Hence, the solution of the recurrence relation is of the form

$$x_n = \begin{cases} nx_1 & ; & \text{if } q = -2, \\ (-1)^{n+1} nx_1 & ; & \text{if } q = 2. \end{cases}$$

This means  $(x_n) \notin \ell_p$ . So, we conclude that there is no eigenvalue in this case.

**Case 2.**  $|q| \neq 2$ .

Then characteristic polynomial has two distinct roots  $|\alpha_1| \neq 1$  and  $|\alpha_2| \neq 1$  with  $\alpha_1 \alpha_2 = 1$ . Let  $|\alpha_2| > 1 > |\alpha_1|$ . The solution of the recurrence relation is of the form

$$x_n = A(\alpha_2)^n + B(\alpha_1)^n.$$

Using the fact that  $qx_1 + x_2 = 0$ , we get  $A = \frac{1}{\alpha_2 - \alpha_1} x_1$ ,  $B = \frac{1}{\alpha_1 - \alpha_2} x_1$ . So we have

$$x_n = \frac{(\alpha_2)^n - (\alpha_1)^n}{\alpha_2 - \alpha_1} x_1.$$

Again we have  $(x_n) \notin \ell_p$ . Hence there is no eigenvalue also in this case.  $\square$

**Theorem 3.4.**  $\sigma_p(U^*(s, r, s), \ell_p^*) = \emptyset$ .

*Proof.* Since  $U^*(s, r, s) = U^t(s, r, s) = U(s, r, s)$  from the Theorem 3.3, the proof is obtained easily.  $\square$

**Corollary 3.5.**  $\sigma_r(U(s, r, s), \ell_p) = \emptyset$ .

**Theorem 3.6.**  $\sigma_c(U(s, r, s), \ell_p) = [r - 2s, r + 2s]$

*Proof.* Since  $\sigma_p(U(s, r, s), \ell_p) = \sigma_r(U(s, r, s), \ell_p) = \emptyset$ ,  $\sigma(U(s, r, s), \ell_p)$  is the disjoint union of the parts  $\sigma_p(U(s, r, s), \ell_p)$ ,  $\sigma_r(U(s, r, s), \ell_p)$  and  $\sigma_c(U(s, r, s), \ell_p)$ , we have  $\sigma_c(U(s, r, s), \ell_p) = [r - 2s, r + 2s]$ .  $\square$

4. THE SPECTRUM OF THE OPERATOR  $U(s, r, s)$  ON THE SEQUENCE SPACE  $bv_p$ ,  $(1 < p < \infty)$

In this section, the fine spectrum of the second order difference operator  $U(s, r, s)$  over the sequence space  $bv_p$ ,  $(1 < p < \infty)$  have been examined. We begin with a theorem concerning the bounded linearity of the operator  $U(s, r, s)$  acting on the sequence space  $bv_p$ ,  $(1 < p < \infty)$ .

**Theorem 4.1.**  $U(s, r, s) \in B(bv_p)$ .

*Proof.* The linearity of the operator  $U(s, r, s)$  is trivial and so it is omitted. Let us take any  $x = (x_k) \in bv_p$ . Then, using Minkowski's inequality and taking the negative indices  $x_{-k} = 0$ , we have

$$\begin{aligned} \|U(s, r, s)x\|_{bv_p} &= \left( \sum_{k=0}^{\infty} |sx_{k-1} + rx_k + sx_{k+1} - (sx_{k-2} + rx_{k-1} + sx_k)|^p \right)^{\frac{1}{p}} \\ &\leq (2|s| + |r|) \|x\|_{bv_p}. \end{aligned}$$

□

**Theorem 4.2.**  $\sigma(U(s, r, s), bv_p) = [r - 2s, r + 2s]$ .

*Proof.* First, we prove that  $(U(s, r, s) - \lambda I)^{-1}$  exists and is in  $B(bv_p)$  for  $\lambda \notin [r - 2s, r + 2s]$  and next that the operator  $(U(s, r, s) - \lambda I)$  is not invertible for  $\lambda \in [r - 2s, r + 2s]$ .

Let  $\lambda \notin [r - 2s, r + 2s]$ . Let  $y = (y_k) \in bv_p$ . This implies that  $(y_k - y_{k-1}) \in \ell_p$ . Solving the equation  $(U(s, r, s) - \lambda I)x = y$ , we find the matrix in the proof of Theorem 3.2. Then we obtain that

$$x_k - x_{k-1} = (U(s, r, s) - \lambda I)^{-1}(y_k - y_{k-1}).$$

Since  $(U(s, r, s) - \lambda I)^{-1} \in (\ell_p, \ell_p)$  by Theorem 3.2,  $(x_k) \in bv_p$ . This shows that  $(U(s, r, s), bv_p) \subseteq [r - 2s, r + 2s]$ .

Now, let  $\lambda \in [r - 2s, r + 2s]$  and  $\lambda \neq r$ . Then  $(U(s, r, s) - \lambda I)^{-1}$  exists. Using (3.4), it can be shown not belong in  $B(\ell_p)$ . If  $\lambda = r$ , then similar arguments as in the proof of Theorem 3.2 show that the operator  $U(s, 0, s) : bv_p \rightarrow bv_p$  is not invertible. This shows that  $[r - 2s, r + 2s] \subseteq \sigma(U(s, r, s), bv_p)$ . This completes the proof. □

Since the spectrum and fine spectrum of the matrix  $U(s, r, s)$  as an operator on the sequence space  $bv_p$  are similar to that of the space  $\ell_p$  in Section 2, to avoid the repetition of the similar statements we give the results in the following theorem without proof.

**Theorem 4.3.** (i)  $\sigma_p(U(s, r, s), bv_p) = \emptyset$ ,  
(ii)  $\sigma_p(U^*(s, r, s), bv_p^*) = \emptyset$ ,



- (iii)  $\sigma_r(U(s, r, s), bv_p) = \emptyset$ ,
- (iv)  $\sigma_c(U(s, r, s), bv_p) = [r - 2s, r + 2s]$ .

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DEPARTMENT OF MATHEMATICAL ENGINEERING, YILDIZ TECHNICAL UNIVERSITY, ESENLER,  
İSTANBUL, TURKEY.

*E-mail address:* `vkaya@yahoo.com`; `vkaya@yildiz.edu.tr`

ADYAMAN UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS,  
ADYAMAN-TURKEY

*E-mail address:* `mmanafov@adyaman.edu.tr`

İSTANBUL COMMERCE UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT  
OF MATHEMATICS, İSTANBUL-TURKEY

*E-mail address:* `necsimsek@yahoo.com`